1. Maximum principle with Neumann boundary condition

Theorem 1. *$u : [-L, L] \times [0, T) \to \mathbb{R}$ is a smooth function satisfying*

$$u_t \leqslant u_{xx}, \qquad \textit{for} \quad |x| \leqslant L, 0 \leqslant t, \tag{1}$$

$$u_x(-L, t) \geqslant 0 \geqslant u_x(L, t), \qquad \textit{for} \quad 0 \leqslant t, \tag{2}$$

$$u(x, 0) \leqslant 0, \qquad \textit{for} \quad |x| \leqslant L. \tag{3}$$

Then, we have

$$u(x, t) \leqslant 0 \tag{4}$$

for all $|x| \leqslant L, 0 \leqslant t$.

Proof. Given $\epsilon > 0$, we define

$$u^\epsilon(x, t) = \epsilon\left[x^2 + 3t + 1\right] \tag{5}$$

Then, we have

$$u^\epsilon_t = 3\epsilon > 2\epsilon = u_{xx}, \qquad \text{for} \quad |x| \leqslant L, 0 \leqslant t, \tag{6}$$

$$u_x(-L, t) = -2L < 0, \;\; u_x(L, t) = 2L > 0, \qquad \text{for} \quad 0 \leqslant t, \tag{7}$$

$$u^\epsilon(x, 0) \geqslant \epsilon > 0, \qquad \text{for} \quad |x| \leqslant L. \tag{8}$$

We claim $u(x, t) < u^\epsilon(x, t)$ holds for all (x, t). Suppose that it fails. Then, since $u^\epsilon(x, 0) > u(x, 0)$, there exists some $(x_0, t_0) \in [-L, L] \times (0, T)$ such that $u(x, t) < u^\epsilon(x, t)$ holds for all $|x| \in L$ and $t \in (0, t_0)$, and we have $u(x_0, t_0) = u^\epsilon(x_0, t_0)$.

Case 1. Suppose $|x_0| < L$.

We define $w(x, t) = u^\epsilon(x, t) - u(x, t)$. Then, $w(x, t) > 0$ for $t < t_0$ implies $w(x, t_0) \geqslant 0$. Moreover, we have $w(x_0, t_0)$ by definition of (x_0, t_0). Namely, $w(x, t_0)$ attains its minimum at the interior point x_0. Hence, we have $w_{xx}(x_0, t_0) \geqslant 0$, namely

$$u^\epsilon_{xx}(x_0, t_0) \geqslant u_{xx}(x_0, t_0). \tag{9}$$

However, $w(x_0, t) > 0 = w(x_0, t_0)$ for $t < t_0$ implies

$$u^\epsilon_t(x_0, t_0) - u_t(x_0, t_0) = w_t(x_0, t_0) = \lim_{t \to t_0} \frac{w(x_0, t_0) - w(x_0, t)}{t_0 - t} \leqslant 0. \tag{10}$$

Therefore, we have a contradiction

$$u_t^\epsilon(x_0,t_0) > u_{xx}^\epsilon(x_0,t_0) \geqslant u_{xx}(x_0,t_0) \geqslant u_t(x_0,t_0) \geqslant u_t^\epsilon(x_0,t_0). \tag{11}$$

Case 2. Suppose $|x_0| = L$.

Without loss of generality, we consider the case $x_0 = L$. We recall $w(x,t_0) \geqslant w(x_0,t_0) = w(L,t_0)$ holds for $|x| \leqslant L$. Therefore,

$$w_x(L,t_0) = \lim_{x\to L} \frac{w(L,t_0) - w(x,t_0)}{L-x} \geqslant 0. \tag{12}$$

This yields a contradiction to the given boundary condition.

$$-2L = u_x^\epsilon(L,t_0) \geqslant u_x(L,t_0) = 0. \tag{13}$$

□

Corollary 2. *$u : [-L,L] \times [0,T) \to \mathbb{R}$ is a smooth function satisfying*

$$u_t \leqslant u_{xx}, \qquad \textit{for} \quad |x| \leqslant L, 0 \leqslant t, \tag{14}$$

$$u_x(-L,t) \geqslant 0 \geqslant u_x(L,t), \qquad \textit{for} \quad 0 \leqslant t, \tag{15}$$

$$u(x,0) = g(x), \qquad \textit{for} \quad |x| \leqslant L. \tag{16}$$

Then, we have

$$u(x,t) \leqslant \max_{|x|\leqslant L} g(x). \tag{17}$$

for all $|x| \leqslant L, 0 \leqslant t$.

Proof. We define $v(x,t) = u(x,t) - \max g$, and apply the maximum principle. □

2. Decay estimate

We begin with establishing a Poincare type inequality.

Lemma 3 (1D Poincare inequality)**.** *Suppose that a smooth function* $u : [0,1] \to \mathbb{R}$ *has a point* $x_0 \in [0,1]$ *satisfying* $u(x_0) = 0$*. Then, the following holds for all* $x \in [0,1]$*.*

$$|u(x)|^2 \leqslant 4 \int_0^1 |u'(s)|^2 ds \tag{18}$$

Proof. Let u attain its maximum at x_1. Without loss of generality, we assume $x_1 \geqslant x_0$.

$$|u(x_1)|^2 = |u(x_1)|^2 - |u(x_0)|^2 = \int_{x_0}^{x_1} \frac{d}{ds}|u(s)|^2 ds = \int_{x_0}^{x_1} 2uu' ds. \tag{19}$$

By the Arithmetic Mean-Geometric Mean inequality, we have

$$\tfrac{1}{2}u^2 + 2|u'|^2 \geqslant 2uu'. \tag{20}$$

Hence,

$$|u(x_1)|^2 \leqslant \int_{x_0}^{x_1} \tfrac{1}{2}u^2 + 2|u'|^2 ds \leqslant \int_0^1 \tfrac{1}{2}u^2 + 2|u'|^2 ds \leqslant \tfrac{1}{2}|u(x_1)|^2 + 2\int_0^1 |u'|^2 ds. \tag{21}$$

Therefore, we obtain the desired result.

$$\tfrac{1}{2}|u(x_1)|^2 \leqslant 2\int_0^1 |u'|^2 ds \tag{22}$$

□

Theorem 4. $u : [0,1] \times [0,T) \to \mathbb{R}$ *is a smooth function satisfying*

$$u_t = u_{xx}, \qquad \textit{for} \quad 0 \leqslant x \leqslant 1, 0 \leqslant t, \tag{23}$$

$$u(0,t) = u(1,t) = 0, \qquad \textit{for} \quad 0 \leqslant t, \tag{24}$$

$$u(x,0) = g(x), \qquad \textit{for} \quad 0 \leqslant x \leqslant 1. \tag{25}$$

Then, we have

$$\int_0^1 |u(x,t)|^2 dx \leqslant e^{-\frac{t}{2}} \int_0^1 |g(x)|^2 dx. \tag{26}$$

Namely, $\lim_{t\to 0} \int u^2 dx = 0$.

Proof. We define an energy

$$E(t) = \int_0^1 |u(x,t)|^2 dx. \tag{27}$$

Then,

$$\frac{d}{dt}E(t) = \int_0^1 2uu_t dx = \int_0^1 2uu_{xx} dx = 2uu_x\big|_0^1 - 2\int_0^1 u_x^2 dx = -2\int_0^1 u_x^2 dx. \tag{28}$$

Since $u(0,t) = u(1,t) = 0$, we can apply the lemma above so that

$$\frac{d}{dt}E(t) \leqslant -\frac{1}{2}\max_{0\leqslant x\leqslant 1}|u(x,t)|^2 = -\frac{1}{2}\int_0^1 \max_{0\leqslant x\leqslant 1}|u(x,t)|^2 dx \leqslant -\frac{1}{2}E(t). \tag{29}$$

Therefore,

$$\frac{d}{dt}\left(e^{\frac{t}{2}}E(t)\right) = e^{\frac{t}{2}}E'(t) + \frac{1}{2}e^{\frac{t}{2}}E(t) \leqslant 0. \tag{30}$$

This gives the desired result.

$$e^{\frac{t}{2}}E(t) \leqslant \int_0^1 g^2(x)dx. \tag{31}$$

□

3. Review: Fourier series

We recall the Fourier series. In this class, we will use the following fact without proofs.

Given a smooth function $f : [-L, L] \to \mathbb{R}$ with $f(-L) = f(L)$, the following holds

$$\lim_{N\to+\infty}\sup_{|x|\leqslant L}|f(x) - S_N(x)| = 0,$$

for the partial sums $S_N(x)$ of Fourier series,

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L) + \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \qquad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx, \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function satisfying $f(0) = 0$. Then,

$$\lim_{N\to+\infty}\sup_{0\leqslant\leqslant L}|f(x) - S_N(x)| = 0,$$

holds for the partial sums $S_N(x)$ of Fourier sine series,

$$S_N(x) = \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),$$

where

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function satisfying $f'(0) = 0$. Then,

$$\lim_{N\to+\infty} \sup_{0\leqslant\leqslant L} |f(x) - S_N(x)| = 0,$$

holds for the partial sums $S_N(x)$ of Fourier cosine series,

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L),$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, \qquad a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx.$$

4. Review: ODE

We recall the some well-known results in ODEs. We will also use them without proofs.

Suppose that a function $u(x)$ satisfies the following differential equation

$$u''(x) + \mu^2 u(x) = 0. \tag{32}$$

Then,

$$u(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x), \tag{33}$$

for some constants c_1, c_2 depending on initial (or boundary data). For example, if $u(x)$ satisfies $u(0) = 0$ and $u'(0) = 1$, then the constants must be $c_1 = \mu^{-1}$ and $c_2 = 0$.

Suppose that a function $u(x)$ satisfies the following differential equation

$$u'(x) = \lambda u(x). \tag{34}$$

Then,

$$u(x) = ce^{\lambda x}, \tag{35}$$

for some constant c depending on the initial data.